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Dynamic Variant of Mathematical Model of Collective Behavior

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Abstract—In 2014, P.S. Krasnoshchekov, Academician at the Russian Academy of Sciences, offered A.A. Belolipetskii to continue research on the collective behavior of people by generalizing his earlier static model to the dynamic case. For this reason, this work is regarded as a tribute to commemorate Krasnoschekov, an outstanding scientist. The fundamental quantitative model Krasnoshchekov proposed in his works studied a static model of collective behavior when people can change their original opinion on a subject after one stage of informational interaction. Opinions are assumed to be alternatives. A person can support his country to join the WTO with probability p and object to it with probability $1 - p$. In this work, multistep opinion exchange processes are considered. Quantitative characteristics of values of probabilities p (of people's opinions) are obtained as functions of the step number and the rate of change of these probabilities. For instance, the way the mass media can control the opinions of their target audience if this audience has certain psychological characteristics is studied.

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INTRODUCTION

The past three decades have seen vigorous growth in research on the collective behavior of people in various situations [1–12]. Some of the works consider the psychological and sociological aspects of collective behavior, while others focus on quantitative estimates that describe the behavior of individuals and groups on the whole.

We briefly describe a static model of collective behavior proposed by Krasnoshchekov in [1–3]. Suppose a decision maker (DM) can make one of two $s = 1, 2$ alternative decisions. The probability that the ith individual, $i = \overline{1, N}$, makes decision number $s = 1$ is $p_i \in [0, 1]$, while it is $1 - p_i$ for $s = 2$.

Suppose μ_i is the individualism coefficient of the ith DM. For $\mu_i = 1$, this DM is absolutely independent and cannot be made to change his/her mind. For $\mu_i = 0$, the DM is an absolute conformist who changes his/her mind to please another opinion. Suppose $\lambda_{ij} \in [0,1]$ is the probability that the ith DM makes decision $s = 1$ after talking to the jth DM given that the jth DM sticks to alternative $s = 1$. Then, for the absolute conformist ($\mu_i = 0$), we can write the probability that after talking to the group he/she makes decision $s = 1$ by the formula of total probability as

$$p_i^0 = \sum_{j=1}^{N} \lambda_{ij} p_j.$$

Obviously, $\lambda_{ii} = 0$ since his/her own opinion has no weight for the conformist and if all $p_j = 1,\ j \neq i$, then $p_i^0 = 1$ (the group will always persuade the conformist to accept the point of view of the "society"). Hence, $\lambda_{i1} + \lambda_{i2} + \ldots \lambda_{iN} = 1,\ i = \overline{1, N}$. If the ith DM is independent in making a decision ($\mu_i = 1$), the respective probability will equal some a priori probability α_i. For the intermediate values $\mu_i \in (0, 1)$, it makes sense to calculate the a posteriori probability in the form of a convex combination

$$p_i = \mu_i \alpha_i + (1 - \mu_i) p_i^0 = \mu_i \alpha_i + (1 - \mu_i) \sum_{j=1}^{N} \lambda_{ij} p_j.$$

1. STATEMENT OF THE PROBLEM AND PROPERTIES OF ITS SOLUTIONS

We generalize the static model described above. If we assume the variables $p_i(t)$ to be functions of continuous time t, we can write the model described above as a system of differential equations

$$\frac{dp_i(t)}{dt} = \mu_i(\alpha_i - p_i(t)) + (1-\mu_i)\sum_{j=1}^{N}\lambda_{ij}\left(p_j(t) - p_i(t)\right).$$

The substantiation and properties of this system are studied in [11]. One can easily see that the stationary solutions of this system satisfy the equations of Krasnoshchekov's static model.

Below, we consider its dynamic aspect. To do this, we put time (step) to be discrete $t_k = k = 0, 1, 2, \ldots$ and assume that the a priori solution at this step equals the a posteriori value obtained at the previous step. For the sake of brevity, we write $p(k)$ instead of $p(t_k)$ in what follows. Now, the model takes the form of a system of linear homogeneous difference equations

$$p_i(k+1) = \mu_i p_i(k) + (1-\mu_i)\sum_{j=1}^{N}\lambda_{i\,j} p_j(k+1), \quad i = \overline{1, N}, \tag{1.1}$$

with the parameters

$$\mu_i \in [0, 1], \quad \lambda_{ij} > 0, \quad i \neq j, \quad \lambda_{ii} = 0, \quad \sum_{j=1}^{N}\lambda_{ij} = 1. \tag{1.2}$$

Find the functions $p_i(k)$ for the given initial conditions $p_i(0)$, $i = \overline{1, N}$.

In [1], two examples (negotiations and elections) were considered, where the particular cases of dynamic relations (1.1) were studied. Below, we obtain the general properties of solutions of system (1.1). Suppose

$$\mathbf{p} = \left(p_1, \ldots, p_N\right)^{\mathrm{T}}, \Lambda = \left(\lambda_{ij}\right), i, j = \overline{1, N}; \quad \mathrm{M} = \mathrm{diag}(\mu_1, \ldots, \mu_N)$$

is a diagonal matrix, E is the unity matrix, and the parameters $\mu_1 + \ldots + \mu_N > 0$. Then, we can write system (1.1) in the vector form

$$(E - A)\mathbf{p}(k+1) = \mathrm{M}\mathbf{p}(k), \tag{1.3}$$

where

$$A = (E - \mathrm{M})\Lambda. \tag{1.4}$$

Definition 1. A quadratic matrix $A\;(N \times N)$ is called decomposable if one can use simultaneous permutation of its rows and columns to represent it in the block-triangular form

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

where $A_1\;(m \times m), A_3\left((N-m)\times(N-m)\right)$, $m < N$.

Proposition 1. If the matrix A is decomposable, the matrix A^2 is decomposable and

$$A^2 = \begin{bmatrix} A_1^2 & \tilde{A}_2 \\ 0 & A_3^2 \end{bmatrix}.$$

The proof follows from Definition 1.

Proposition 2. If all $\mu_i < 1$, matrix (1.4) is indecomposable.

Proof. Suppose $a_{ij}^{(2)}$ is an arbitrary element of the matrix A^2 that is at the intersection of the ith row and jth column. It follows from (1.4) that

$$a_{ij}^{(2)} = \sum_{k=1}^{N}(1-\mu_i)\lambda_{ik}\left(1-\mu_k\right)\lambda_{kj}.$$

It follows from (1.2) and the proposition hypothesis that $a_{ij}^{(2)} > 0$. This and Proposition 1 leads to the validity of the proposition.

Theorem 1 (Perron–Frobenius). If A is indecomposable and nonnegative (i.e., all its elements are nonnegative), it has the eigenvalue λ_A such that for any other eigenvalue λ $|\lambda| < \lambda_A$ holds and the respective right eigen column vector $\mathbf{x}_A$ $(A\mathbf{x}_A = \lambda_A\mathbf{x}_A)$ and the left eigen row vector $\mathbf{y}_A$ $(\mathbf{y}_A A = \lambda_A \mathbf{y}_A)$ are positive (i.e., their components are positive).

For the proof of the theorem, see, for instance, [13, 14].

Definition 2. The eigenvalue λ_A is called the Frobenius number and the vectors $\mathbf{x}_A$ and $\mathbf{y}_A$ are called the right and left Frobenius vectors.

Suppose

$$r = \min_{i=\overline{1,N}} r_i, \quad R = \max_{i=\overline{1,N}} r_i, \quad \text{where} \quad r_i = \sum_{j=1}^{N} a_{ij}.$$

Proposition 3. If A is indecomposable and nonnegative, its Frobenius number $r < \lambda_A < R$ for $r < R$. If $r = R$, then $\lambda_A = R$.

One can find the proof in [14].

Definition 3. We call the nonnegative matrix A productive if there exists a nonnegative matrix $(E - A)^{-1}$.

Proposition 4. A nonnegative indecomposable matrix A is productive if and only if its Frobenius number $\lambda_A < 1$.

For the proof, see [14].

Proposition 5. If $\mu_1 + \ldots + \mu_N > 0$, the Frobenius number of matrix (1.4) is less than the unity.

Proof. The sum of the elements of the ith row of matrix (1.4) is $r_i = (1 - \mu_i)(\lambda_{i1} + \lambda_{i2} + \ldots \lambda_{iN}) = (1 - \mu_i) \le 1$. Hence,

$$r = \min_{i=\overline{1,N}} r_i > 0, \quad R = \max_{i=\overline{1,N}} r_i \le 1.$$

If $r < R$, it follows from Proposition 3 that $\lambda_A < R \le 1$. If $r = R$, then all $r_i = R = 1 - \mu < 1$. Then, the equality $\lambda_A = R < 1$ follows from Proposition 1.3.

Lemma 1. If all parameters $\mu_1 + \ldots + \mu_N > 0$, system of equations (1.3) and, hence, system (1.1) are unambiguously solvable and the recurrent relations hold

$$\mathbf{p}(k+1) = (E - A)^{-1} \mathrm{M} \mathbf{p}(k). \tag{1.5}$$

If, in addition, $\mathbf{p}(k) \ge 0$, then $\mathbf{p}(k+1) \ge 0$.

Proof. It follows from the hypotheses of this lemma and Proposition 2 that the nonnegative matrix A is indecomposable, and we can conclude from Proposition 5 that its Frobenius number λ_A is less than unity. Then, Proposition 4 ensures the matrix $E - A$ is productive, i.e., Eq. (1.3) has solution (1.5).

For arbitrary $k = 0, 1, 2, \ldots$, we denote

$$\pi_{\max}(k) = \max_{1 \le j \le N} p_j(k), \quad \pi_{\min}(k) = \min_{1 \le j \le N} p_j(k).$$

Lemma 2. For any positive matrix $\Lambda = (\lambda_{ij})$, $i, j = \overline{1, N}$, with the sum of its elements in any row equaling the unity and all its diagonal elements being zero, and the values of the parameters $\mu_1, \ldots, \mu_N \in [0, 1]$, $\mu_1 + \ldots + \mu_N > 0$, the solutions of Eq. (1.1) are such that the following inequalities hold:

$$\pi_{\max}(k+1) \le \pi_{\max}(k), \quad \pi_{\min}(k+1) \ge \pi_{\min}(k). \tag{1.6}$$

Moreover, there exists $\varepsilon_0 \in (0, 1)$ such that the following inequality holds:

$$\pi_{\max}(k) - \pi_{\min}(k) \le \varepsilon_0^k (\pi_{\max}(0) - \pi_{\min}(0)). \tag{1.7}$$

Proof. 1. If $\pi_{\max}(k) = \pi_{\min}(k) = p$, then all $p_i(k) = \pi_{\max}(k) = \pi_{\min}(k) = p$, $i = \overline{1, N}$. In this case, all $p_i(k+1) = p$, $i = \overline{1, N}$. One can easily check the latter by substituting the values $p_i(k+1) = p_i(k) = p$ to Eqs. (1.1). By Lemma 1, this system is unambiguously solvable, i.e., $p_i(k+1) = p$ is the unique solution. Hence, the lemma is obvious.

2. Now, we put $\pi_{\max}(k+1) > \pi_{\min}(k+1)$. Suppose $p_m(k+1) = \pi_{\max}(k+1)$. We can take $\mu_m > 0$. Indeed, if $\mu_m = 0$, we have the equality

$$\pi_{\max}(k+1) = \sum_{j=1}^{N} \lambda_{m\,j} p_j(k+1)$$

from (1.1). By the properties of the matrix Λ, the latter is possible only if all $p_j(k+1) = \pi_{\max}(k+1)$, which contradicts our assumption. Further, suppose $p_h(k+1) = \pi_{\min}(k+1)$. As we showed above, we can take $\mu_h > 0$. Then, we have from (1.1)

$$\begin{aligned}
\mu_m p_m(k) &= \pi_{\max}(k+1) - (1-\mu_m)\sum_{j=1}^{N}\lambda_{m\,j}p_j(k+1) = \mu_m\pi_{\max}(k+1) \\
&+ (1-\mu_m)\sum_{j=1}^{N}\lambda_{m\,j}\left(\pi_{\max}(k+1) - p_j(k+1)\right) \\
&= \mu_m\pi_{\max}(k+1) + (1-\mu_m)\lambda_{m\,h}\left[\pi_{\max}(k+1) - \pi_{\min}(k+1)\right] \\
&+ (1-\mu_m)\sum_{j=1,\, j\neq h}^{N}\lambda_{m\,j}\left(\pi_{\max}(k+1) - p_j(k+1)\right) \\
&\geq \mu_m\pi_{\max}(k+1) + (1-\mu_m)\lambda_{m\,h}\left[\pi_{\max}(k+1) - \pi_{\min}(k+1)\right].
\end{aligned}$$

This and the inequality $p_m(k) \leq \pi_{\max}(k)$ lead to

$$\pi_{\max}(k) - \pi_{\max}(k+1) \geq \frac{(1-\mu_m)\lambda_{m\,h}}{\mu_m}\left[\pi_{\max}(k+1) - \pi_{\min}(k+1)\right]. \tag{1.8}$$

Hence, $\pi_{\max}(k+1) < \pi_{\max}(k)$.

We estimate

$$\begin{aligned}
\mu_h p_h(k) &= \pi_{\min}(k+1) - (1-\mu_h)\sum_{j=1}^{N}\lambda_{h\,j}p_j(k+1) \\
&= \mu_h\pi_{\min}(k+1) + (1-\mu_h)\sum_{j=1}^{N}\lambda_{h\,j}(\pi_{\min}(k+1) - p_j(k+1)) \\
&= \mu_h\pi_{\min}(k+1) + (1-\mu_h)\lambda_{h\,m}\left[\pi_{\min}(k+1) - \pi_{\max}(k+1)\right] \\
&+ (1-\mu_h)\sum_{j=1,\, j\neq m}^{N}\lambda_{h\,j}(\pi_{\min}(k+1) - p_j(k+1)) \\
&\leq \mu_h\pi_{\min}(k+1) + (1-\mu_h)\lambda_{h\,m}\left[\pi_{\min}(k+1) - \pi_{\max}(k+1)\right].
\end{aligned}$$

This and the inequality $p_h(k) \geq \pi_{\min}(k)$ lead to

$$\pi_{\min}(k+1) - \pi_{\min}(k) \geq \frac{(1-\mu_h)\lambda_{h\,m}}{\mu_h}\left[\pi_{\max}(k+1) - \pi_{\min}(k+1)\right]. \tag{1.9}$$

Hence, $\pi_{\min}(k+1) - \pi_{\min}(k) > 0$. Inequalities (1.6) are proved. We summarize the left- and right-hand sides of inequalities (1.8) and (1.9)

$$\begin{gathered}(\pi_{\max}(k) - \pi_{\min}(k)) - (\pi_{\max}(k+1) - \pi_{\min}(k+1)) \\ \geq \left[\frac{(1-\mu_m)\lambda_{mh}}{\mu_m} + \frac{(1-\mu_h)\lambda_{hm}}{\mu_h}\right][\pi_{\max}(k+1) - \pi_{\min}(k+1)] \\ \geq 2c[\pi_{\max}(k+1) - \pi_{\min}(k+1)],\end{gathered}$$

where

$$c = \min_{\mu_i > 0} \frac{(1-\mu_i)}{\mu_i} \cdot \min_{i \neq j} \lambda_{ij} > 0.$$

Hence, $(\pi_{\max}(k) - \pi_{\min}(k)) \geq (1+2c)[\pi_{\max}(k+1) - \pi_{\min}(k+1)]$ or $(\pi_{\max}(k+1) - \pi_{\min}(k+1)) \leq \varepsilon[\pi_{\max}(k) - \pi_{\min}(k)]$, where $\varepsilon = (1+2c)^{-1} < 1$. The latter inequality ensures estimate (1.7) holds. The lemma is proved.

Corollary 1. All values $p_i(k) \xrightarrow[k\to\infty]{} p^*$, $i = \overline{1, N}$.

Theorem 2. If the hypotheses of Lemma 2 hold and all components $p_i(0) \in [a, b]$, then all components $p_i(k) \in [a, b]$, $k = 1, 2, \dots$.

The proof follows from Lemma 2. In particular, the following corollaries hold.

Corollary 2. If all components $p_i(0) \in [0, 1]$, then all components $p_i(k) \in [0, 1]$, $k = 1, 2, \dots$.

Corollary 3. In the one-step model, where $p_i(0) = \alpha_i \in [0, 1]$, $p_i(1) = p_i$, $i = \overline{1, N}$, all values $p_i \in [0, 1]$.

We study if system of difference equations (1.1) is stable to the perturbation of the initial conditions. Suppose, $p_i(k)$ are the solutions of system (1.1) for the given initial conditions $p_i(0)$, $i = \overline{1, N}$, and $p_i(k) + \delta p_i(k)$ are the solutions of system (1.1) for the perturbed initial conditions $p_i(0) + \delta p_i(0)$, $i = \overline{1, N}$.

Theorem 3 (on stability). The solutions of system of equations (1.1) are stable to perturbations of the initial conditions. In other words, if all perturbations $\delta p_i(0) \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, then all variations $\delta p_i(k) \in [-\varepsilon, \varepsilon]$, $i = \overline{1, N}$, $k = 1, 2, \dots$.

Proof. Since relations (1.1) are linear, the equations for the variations $\delta p_i(k)$ have the same form as (1.1), i.e.,

$$\delta p_i(k+1) = \mu_i \delta p_i(k) + (1-\mu_i)\sum_{j=1}^{N} \lambda_{ij}\delta p_j(k+1), \quad i = \overline{1, N}.$$

This and Theorem 2 lead to the hypothesis of Theorem 3.

We consider two matrices $C = (c_{ij})$, $D = (d_{ij})$ (both of dimension $N \times N$) and the complex number z. Suppose $C(z) = (c_{ij}z^{\varsigma_{ij}})$, $D(z) = (d_{ij}z^{\xi_{ij}})$, where the parameters ς_{ij}, ξ_{ij} can be zero or unity only. We search for the nontrivial solution of the homogeneous system of linear equations

$$(C(z) - D(z))\mathbf{x} = 0. \tag{1.10}$$

This solution exists if and only if

$$\det(C(z) - D(z)) = 0. \tag{1.11}$$

Characteristic equation (1.11) is an algebraic equation of a degree not higher than N. We call the root z^* of Eq. (1.11) the generalized characteristic number of Eq. (1.10) and its respective solution $\mathbf{x}^*$ the characteristic vector of this equation. Obviously, the characteristic vector is given up to the accuracy of a factor.

Theorem 4. If the sum of the elements of any row of the matrix $C(1)$ equals the sum of elements of the same row of the matrix $D(1)$, Eq. (1.11) has the root $z^* = 1$ with its respective characteristic vector $\mathbf{x}^*$, all components of which are unities.

Proof. Under our assumptions, the sum of the columns of the matrix $C(1) - D(1)$ is zero, i.e., $\det(C(1) - D(1)) = 0$. Hence, $z^* = 1$ is the generalized characteristic number of Eq. (1.10). We put $\mathbf{x}^* = (1, 1, \ldots, 1)^{\mathrm{T}}$. Then, we can write any of the equations of system (1.10) as

$$\sum_{j=1}^{N}(c_{ij} - d_{ij})x_j^* = \sum_{j=1}^{N}(c_{ij} - d_{ij}) = 0,$$

i.e., $\mathbf{x}^*$ is the characteristic vector that corresponds to the generalized characteristic number $z^* = 1$. The theorem is proved.

We consider the matrix

$$G(z) = (E - A)z - \mathrm{M}. \tag{1.12}$$

Lemma 3. Matrix (1.2) has the generalized characteristic number $z = 1$ and its respective characteristic vector $\mathbf{x} = (1, 1, \ldots, 1)^{\mathrm{T}}$. For any number z, the matrices $C(z) = (E - A)z$ and $D(z) = \mathrm{M}$ satisfy the hypotheses of Theorem 4.

Proof. The sum of the elements of the ith row of the matrix $C(z) = (E - A)z$, by (1.4), is

$$z - z(1 - \mu_i)\sum_{j=1}^{N}\lambda_{ij} = z - z(1 - \mu_i) = z\mu_i,$$

and the respective sum of the elements of the matrix $D(z) = \mathrm{M}$ is μ_i. For $z = 1$, they coincide. Hence, Lemma 3 follows from Theorem 4.

Theorem 5. Suppose z is the generalized characteristic number and $\mathbf{x}$ is its respective characteristic vector of matrix (1.12). Then, the function $\mathbf{p}(k) = z^k\mathbf{x}$ is the solution of Eq. (1.3).

Proof. Substituting the expression $\mathbf{p}(k) = z^k\mathbf{x}$ into Eq. (1.3), we have the system of equations $G(z)\mathbf{x} = 0$. Now, the proposition follows from the hypothesis of this theorem.

Theorem 6. All real generalized characteristic numbers of the matrix $G(z) = (E - A)z - \mathrm{M}$ are not greater than unity in modulo and its characteristic vectors corresponding to the characteristic numbers $z \neq 1$ must have either components of the opposite signs or zero components.

Proof. 1. Suppose $z > 0$. Since the real characteristic vector $\mathbf{x}$ is nontrivial and given up to the accuracy of a factor, we take its maximal component to be positive. Suppose it is x_1. Without loss of generality, its minimal component is x_2. By Theorem 5, $\mathbf{p}(k) = z^k\mathbf{x}$. We consider the vectors $\mathbf{p}(0) = \mathbf{x}$, $\mathbf{p}(1) = z\mathbf{x}$. Obviously, their first and second components are maximal and minimal, respectively. By Lemma 2, $p_1(1) = zx_1 \le p_1(0) = x_1$. This leads to $z < 1$. The same lemma leads to $p_2(1) = zx_2 \ge p_2(0) = x_2$. This and the inequality $z < 1$ leads to $x_2 \le 0$.

2. If $z < 0$, the maximal component of the vector $\mathbf{p}(1)$ is zx_2 and its minimal component is zx_1. By Lemma 2, we have $x_1 \ge zx_2$, $x_2 \le zx_1$. Since $x_1 > 0$, $z < 0$, both inequalities lead to $x_2 \le 0$. If $x_2 \neq 0$, both inequalities lead to $z \ge x_2/x_1$, $z \ge x_1/x_2$. Since either $z \ge x_2/x_1 > -1$, or $x_1/x_2 \ge -1$, then $z \ge -1$.

Theorem 7. If all $\mu_i < 1$, $i = \overline{1, N}$, all generalized characteristic numbers and characteristic vectors of matrix (1.12) are eigenvalues and eigenvectors of the matrix

$$L = (E - A)^{-1}\mathrm{M}. \tag{1.13}$$

The proof follows from Lemma 1.

Theorem 8. If

$$\sum_{i=1}^{N}\mu_i > 0, \quad \mu_i \in [0, 1),$$

the value $z = -1$ is not a generalized characteristic number of matrix (1.12) and, hence, it is not an eigenvalue of matrix (1.13).

Proof. We assume the opposite. Then, for the respective characteristic vector $\mathbf{x}$, we have $G(-1)\mathbf{x} = \mathbf{0}$ or

$$-x_i + (1-\mu_i)\sum_{j=1}^{N}\lambda_{ij}x_j - \mu_i x_i = 0, \quad \overline{i = 1, N}. \tag{1.14}$$

Since $\mathbf{x}$ is a nontrivial real vector, we can assume that its maximal coordinate is positive. By reasoning similar to that used in the proof of Theorem 2, we can show that among maximal coordinates one can find a coordinate with number m such that $\mu_m > 0$. Majorizing the left-hand side of (1.14), we have

$$-x_m + (1-\mu_m)x_m\sum_{j=1}^{N}\lambda_{mj} - \mu_m x_m \ge -x_m + (1-\mu_m)\sum_{j=1}^{N}\lambda_{mj}x_j - \mu_m x_m = 0. \quad \text{Or} \quad 2\mu_m x_m \le 0.$$

The latter is impossible since $x_m > 0$.

Assumption. The multiplicity of all eigenvalues of matrix (1.13) is unity.

If it is true, they all, together with the vector $\mathbf{x}^* = (1, 1, \ldots 1)^{\mathrm{T}}$, form the basis in the N-dimensional complexified space. Suppose these eigenvalues are $z_1 = 1, z_2, z_3, \ldots, z_N$ and their respective eigenvectors are $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N$, and $\mathbf{x}_1 = \mathbf{x}^* = (1, 1, \ldots 1)^{\mathrm{T}}$. Then, the general solution of Eq. (1.3) takes the form

$$\mathbf{p}(k) = y_1\mathbf{x}^* + \sum_{i=2}^{N} y_i\mathbf{x}_i z_i^k, \tag{1.15}$$

where y_i are arbitrary constants. They are given by the initial conditions

$$\mathbf{p}(0) = y_1\mathbf{x}^* + \sum_{i=2}^{N} y_i\mathbf{x}_i. \tag{1.16}$$

System of equations (1.16) is solvable due to the reasoning given above. If all eigenvalues z_i, $i = \overline{1, N}$, of matrix (1.13) are real, Theorems 6–8 lead to $|z_i| < 1$, $\overline{i = 2, N}$. This and (1.15) lead to $\lim_{k\to\infty}\mathbf{p}(k) = y_1\mathbf{x}^*$.

2. MODIFICATION OF THE MODEL FOR THE CASE OF DIFFERENT-SCALE REACTION TIMES OF GROUP PARTICIPANTS CHANGING THEIR OPINIONS

We write (1.1) as

$$p_i(k+1) - p_i(k) = (1-\mu_i)\sum_{j=1}^{N}\lambda_{ij}\,[p_j(k+1) - p_i(k)], \quad i = \overline{1, N}. \tag{2.1}$$

We assume that different members of the group react to a change in their opinion at a different pace. In other words, there exists at least one individual with the maximal reaction speed, which we take to be unity. For other members, their individual speed is $\tau_i \in [0, 1]$. Now, (2.1) takes the form

$$p_i(k+1) - p_i(k) = \tau_i(1-\mu_i)\sum_{j=1}^{N}\lambda_{ij}\,[p_j(k+1) - p_i(k)], \quad i = \overline{1, N}. \tag{2.2}$$

Theorem 9. We can write system of equations (2.2) as

$$p_i(k+1) = \tilde{\mu}_i p_i(k) + (1-\tilde{\mu}_i)\sum_{j=1}^{N}\lambda_{ij}p_j(k+1), \quad i = \overline{1, N}, \tag{2.3}$$

where $\tilde{\mu}_i(\mu_i, \tau_i) = 1 - \tau_i(1-\mu_i) \in [0, 1]$. Obviously, $\tilde{\mu}_i(0, \tau_i) = 1 - \tau_i$, $\tilde{\mu}_i(1, \tau_i) = 1$, $\tilde{\mu}_i(\mu, 0) = 1$, $\tilde{\mu}_i(\mu, 1) = \mu$.

Proof. We rewrite (2.2) as

$$\begin{aligned} p_i(k+1) &= p_i(k) + \tau_i(1-\mu_i)\left[\sum_{j=1}^{N}\lambda_{i\,j}\ p_j(k+1) - \sum_{j=1}^{N}\lambda_{i\,j}\ p_i(k)\right] \\ &= p_i(k+1) = p_i(k) + \tau_i(1-\mu_i)\left[\sum_{j=1}^{N}\lambda_{i\,j}\ p_j(k+1) - p_i(k)\right] \\ &= [1-\tau_i(1-\mu_i)]p_i(k) + \tau_i(1-\mu_i)\sum_{j=1}^{N}\lambda_{i\,j}\ p_j(k+1). \end{aligned} \tag{2.4}$$

Here, we used equality (1.2). We put $\tilde{\mu}_i(\mu_i, \tau_i) = 1 - \tau_i(1-\mu_i)$. Then,

$$(1-\mu_i) = \frac{1-\tilde{\mu}_i}{\tau_i}.$$

We substitute these expressions into (2.4) and obtain what we sought.

It follows from Theorem 9 that Eq. (2.3) have the same form (1.1) even for different reaction pace of individuals to the changes. Hence, this model also preserves the properties of the solutions described in Section 1.

3. PROBLEM OF COMPETITION OF TWO PARLIAMENTARY GROUPS

Within the theory described above, we consider the following hypothetical problem. Suppose the parliament of a country is discussing the enactment of a law that is important for the people of the country, for instance, the British parliament is discussing Brexit. We assume that the members of parliament are divided into two groups. One is mostly for enactment, while the other is against it. We call the first group *Labor* and the second group *Tories*. The parliament holds debates on this issue that can consist of several stages. These debates can change the opinions of the group members. The question is how long it will take the members of parliament to make the final decision and what their decision is going to be.

To obtain a mathematically significant answer, we divide the group of N parliamentarians into two subgroups—in the first one, $i = \overline{1, N_1}$, the group members have $\mu_i = \mu_0$ (Labor); in the second one, $s = N_1 + 1, \ldots, N$, they have $\mu_s = \mu_{00}$ (Tories). The factors of influence of members of both groups on each other are

(1) for all $i = \overline{1, N_1}$, $\lambda_{ii} = 0$

$$\lambda_{ij} = \frac{1-\Lambda_0}{N_1 - 1}, \quad j = \overline{1, N_1}, \quad \lambda_{ij} = \frac{\Lambda_0}{N - N_1}, \quad j = \overline{N_1 + 1, N}; \tag{3.1}$$

(2) for all $s = \overline{N_1 + 1, N}$, $\lambda_{ss} = 0$

$$\lambda_{sj} = \frac{\Lambda_{00}}{N_1}, \quad j = \overline{1, N_1}, \quad \lambda_{sj} = \frac{1-\Lambda_{00}}{N - N_1 - 1}, \quad j = \overline{N_1 + 1, N}. \tag{3.2}$$

The coefficients $\mu_0, \mu_{00}, \Lambda_0, \Lambda_{00} \in [0, 1]$. Moreover, Λ_0 is the factor of influence of the members of the second group on the opinion of the members of the first group. In other words, the higher the value of Λ_0 the higher the probability that a member of the second group will persuade a member of the first group to his/her opinion. Similarly, Λ_{00} is the factor of influence of the members of the first group on the opinion of the members of the second group. The following proposition is obvious.

Proposition 6. In the nonnegative matrix $\Lambda = (\lambda_{pr})$, $p, r = \overline{1, N}$, the sum of the elements in any row is unity and all its diagonal elements are zero.

We write the difference equations (3.1) and (3.2) as

$$\begin{aligned} p_i(k+1) &= \mu_0 p_i(k) + (1-\mu_0)\frac{1-\Lambda_0}{N_1-1}\sum_{j=1}^{N_1}(1-\delta_{ij})p_j(k+1) \\ &+ (1-\mu_0)\frac{\Lambda_0}{N-N_1}\sum_{j=N_1+1}^{N} p_j(k+1),\ i=\overline{1, N_1}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} p_s(k+1) &= \mu_{00} p_s(k) + (1-\mu_{00})\frac{\Lambda_{00}}{N_1}\sum_{j=1}^{N_1} p_j(k+1) \\ &+ (1-\mu_{00})\frac{1-\Lambda_{00}}{N-N_1-1}\sum_{j=N_1+1}^{N}(1-\delta_{sj})p_j(k+1), \quad s=\overline{N_1+1, N}. \end{aligned} \tag{3.4}$$

Below, we consider the following conditions to hold: the parameters $\mu_0, \mu_{00} \in (0, 1)$ and$\Lambda_0, \Lambda_{00} \in (0, 1)$. Then, the system of difference equations (3.3)–(3.4) satisfies the hypotheses of Lemma 1, and, hence, it has the unique solution for the given initial conditions.

To obtain analytical results, we first assume that all

$$p_i(0) = \alpha_0, \quad i=\overline{1, N_1}, \quad p_s(0) = \alpha_{00}, \quad s=\overline{N_1+1, N}. \tag{3.5}$$

We show that the solution of (3.3) and (3.4) has the form

$$p_i(k) = P_1(k), \quad i=\overline{1, N_1}, \quad p_s(k) = P_2(k), \quad s=\overline{N_1+1, N}. \tag{3.6}$$

Indeed, we will search for the solution in form (3.6) with initial conditions (3.5). The equations for P_1, P_2 follow immediately from (3.3)–(3.5)

$$P_1(k+1) = \mu_0 P_1(k) + (1-\mu_0)(1-\Lambda_0)P_1(k+1) + (1-\mu_0)\Lambda_0 P_2(k+1), \tag{3.7}$$

$$P_2(k+1) = \mu_{00} P_2(k) + (1-\mu_{00})\Lambda_{00} P_1(k+1) + (1-\mu_{00})(1-\Lambda_{00})P_2(k+1), \tag{3.8}$$

$$P_1(0) = \alpha_0, \quad P_2(0) = \alpha_{00}. \tag{3.9}$$

We can write them in the matrix form as system (1.3)–(1.4), where the matrix

$$A = \begin{pmatrix} (1-\mu_0)(1-\Lambda_0) & (1-\mu_0)\Lambda_0 \\ (1-\mu_{00})\Lambda_{00} & (1-\mu_{00})(1-\Lambda_{00}) \end{pmatrix}$$

and, hence, system of equations (3.7)–(3.9) can be unambiguously solved. In other words, the solution of (3.6) with initial conditions (3.9) is the solution of system (3.3)–(3.4) with initial conditions (3.5). Since the solution to problem (3.3)–(3.5) is unique, the found solution (3.6) is the sought one.

We search specific solutions of system (3.7) and (3.8) in the form

$$\begin{pmatrix} P_1(k) \\ P_2(k) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} z^k.$$

To find z, $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$, we obtain Eqs. (1.10) and (1.11), where

$$C(z) = \begin{pmatrix} [(1-\mu_0)(1-\Lambda_0)-1]z & (1-\mu_0)\Lambda_0 z \\ (1-\mu_{00})\Lambda_{00} z & [(1-\mu_{00})(1-\Lambda_{00})-1]z \end{pmatrix}, \quad D(z) = \begin{pmatrix} -\mu_0 & 0 \\ 0 & -\mu_{00} \end{pmatrix}. \tag{3.10}$$

In what follows, we use the designation $C = C(1)$. The form of the matrices C, D leads to $c_{11} + c_{12} = -\mu_0$, $c_{21} + c_{22} = -\mu_{00}$. In other words, C, D satisfy the hypotheses of Theorem 4; therefore, one of the roots of the characteristic equation

$$z^2 \det C + z(c_{11}\mu_{00} + c_{22}\mu_0) + \mu_0\mu_{00} = 0, \tag{3.11}$$

$z_1 = 1$, and its respective solution has the form $\mathbf{x} = (x_{11}, x_{12})^{\mathrm{T}} = (1, 1)^{\mathrm{T}}$.

We find the second characteristic number. By Vieta's theorem, the second root of Eq. (3.11) is $z_2 = \mu_0\mu_{00}/\det C$. Since $c_{11} = -c_{12} - \mu_0$ and $c_{22} = -c_{21} - \mu_{00}$, $\det C = (c_{12}\mu_{00} + c_{21}\mu_0) + \mu_0\mu_{00} \geq \mu_0\mu_{00} \geq 0$. If we exclude the case $\mu_0 = \mu_{00} = 0$, it leads to the expression $z_2 = \mu_0\mu_{00}/(c_{12}\mu_{00} + c_{21}\mu_0 + \mu_0\mu_{00}) \in [0, 1]$. One can easily see that $z_2 \in (0, 1)$ if $\mu_0\mu_{00} > 0$ and either $\mu_0 < 1$, $\Lambda_0 > 0$ or $\mu_{00} < 1$, $\Lambda_{00} > 0$.

In what follows, we consider these conditions to hold. The characteristic vector associated with the generalized characteristic number z_2 has the form

$$\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ \dfrac{\mu_0}{c_{12}}\left(\dfrac{1}{z_2} - 1\right) \end{pmatrix} = \begin{pmatrix} -1 \\ \dfrac{\mu_0(1-\mu_{00})\Lambda_{00}}{\mu_{00}(1-\mu_0)\Lambda_0} \end{pmatrix}$$

up to the accuracy of a factor. Then, we can write the general solution of system (3.7) and (3.8) as

$$\begin{pmatrix} P_1(k) \\ P_2(k) \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta_2 \begin{pmatrix} -1 \\ \dfrac{\mu_0(1-\mu_{00})\Lambda_{00}}{\mu_{00}(1-\mu_0)\Lambda_0} \end{pmatrix} z_2^k. \tag{3.12}$$

We find constants β_1, β_2 from initial conditions (3.9). They are

$$\beta_1 = \alpha_0 + (\alpha_{00} - \alpha_0)/(1+\xi), \quad \beta_2 = (\alpha_{00} - \alpha_0)/(1+\xi), \tag{3.13}$$

where $\xi = (\mu_0(1-\mu_{00})\Lambda_{00})/(\mu_{00}(1-\mu_0)\Lambda_0) \geq 0$. It follows from (3.12) and (3.13) that for $\alpha_{00} > \alpha_0$ the function $P_1(k)$ is monotonically decreasing and $P_2(k)$ is monotonically increasing, and vice versa for $\alpha_{00} < \alpha_0$. For $k \to \infty$, both functions take equivalent values that are $\beta_1 = \alpha_0 + (\alpha_{00} - \alpha_0)/(1+\xi)$. If $\alpha_{00} = \alpha_0 = \alpha$, then $P_1(k) = P_2(k) \equiv \alpha$. Thus, the value

$$P(\infty) = P_1(\infty) = P_2(\infty) = \beta_1$$
$$= \frac{\mu_0(1-\mu_{00})\Lambda_{00}}{\mu_0(1-\mu_{00})\Lambda_{00} + \mu_{00}(1-\mu_0)\Lambda_0}\alpha_0 + \frac{\mu_{00}(1-\mu_0)\Lambda_0}{\mu_0(1-\mu_{00})\Lambda_{00} + \mu_{00}(1-\mu_0)\Lambda_0}\alpha_{00};$$

i.e., it is the convex combination of the initial values α_0, α_{00}.

The mean value of the members of both groups adhering to Labour's point of view is

$$M(\infty) = NP(\infty)$$
$$= N\left[\frac{\mu_0(1-\mu_{00})\Lambda_{00}}{\mu_0(1-\mu_{00})\Lambda_{00} + \mu_{00}(1-\mu_0)\Lambda_0}\alpha_0 + \frac{\mu_{00}(1-\mu_0)\Lambda_0}{\mu_0(1-\mu_{00})\Lambda_{00} + \mu_{00}(1-\mu_0)\Lambda_0}\alpha_{00}\right].$$

Suppose $\alpha_0 = 1$, $\alpha_{00} = 0$. Then,

$$M(\infty) = N\frac{\mu_0(1-\mu_{00})\Lambda_{00}}{\mu_0(1-\mu_{00})\Lambda_{00} + \mu_{00}(1-\mu_0)\Lambda_0}.$$

If $\mu_{00} = 1 - \varepsilon$ and μ_0 is a fixed variable, then $M(\infty) = \dfrac{N\varepsilon\mu_0\Lambda_{00}}{\varepsilon\mu_0\Lambda_{00} + (1-\varepsilon)(1-\mu_0)\Lambda_0} \xrightarrow[\varepsilon\to o]{} 0$; i.e., if the Tories do not feel like compromising, then, regardless of the other fixed parameters, they will persuade the opposite party to their opinion ($P = 0$). If also $\mu_0 = 1 - \varepsilon$, then $M(\infty) = N\Lambda_{00}/(\Lambda_{00} + \Lambda_0)$. Then, the more convincing group wins. For instance, for $\Lambda_0 > \Lambda_{00}$ (the Tories are more convincing than Labor), the value $M(\infty) < N/2$, while the value $M(\infty) \geq N/2$ for $\Lambda_0 \leq \Lambda_{00}$.

We assume that

$$\begin{aligned} p_i(0) &= \alpha_0 + \varepsilon\,\delta p_i(0) \in [0, 1], \quad i = \overline{1, N_1}, \\ p_s(0) &= \alpha_{00} + \varepsilon\,\delta p_s(0) \in [0, 1], \quad s = \overline{N_1 + 1, N}. \end{aligned} \tag{3.14}$$

We denote the vectors

$$\mathbf{P}(k) = \Big(\underbrace{P_1(k), \ldots, P_1(k)}_{N_1}, \underbrace{P_2(k), \ldots, P_2(k)}_{N-N_1}\Big)^{\mathrm{T}}, \quad \delta\mathbf{P}(k) = (\delta p_1(k), \ldots, \delta p_N(k))^{\mathrm{T}}.$$

Theorem 10. If all $|\delta p_j(0)| < 1$, the solution of system (3.3) and (3.4) with initial conditions (3.14) has the form $\mathbf{P}(k) + \varepsilon\, \delta\mathbf{P}(k) \in [0, 1]$, with all components being $|\delta p_i(k)| < 1$.

The proof follows from Theorem 2.

4. PROBLEM OF PROPHETS AND FALSE PROPHETS

In the problem we consider in this section, we obtain the quantitative characteristics of the influence of two mass media units on the public. These mass media units are considered to adhere to two opposite points of view on some issue and stand firm on their beliefs, such as the mass media supporting candidates of the Democratic or Republican Parties in the US election campaign. We call one mass media unit a Prophet and the other one a False Prophet. Suppose

$$\lambda_{ij} = \lambda = \frac{1 - \Lambda_0 - \Lambda_{00}}{N - 1}, \quad i \neq j, \quad \lambda_{ii} = 0, \quad \Lambda_0 \geq 0, \quad \Lambda_{00} \geq 0, \quad \Lambda_0 + \Lambda_{00} < 1 \tag{4.1}$$

for N people. We add two more participants, i.e., a Prophet with $\alpha_0 = 1, \mu_0 = 1$ and number 0 and a False Prophet with $\alpha_{00} = 0, \mu_{00} = 1$ and number 00. The other participants have values $\alpha_i \in (0, 1), \mu_i = \mu \in (0, 1), i = 1, 2, \ldots, N$. We consider the process to be evolving in time and $\alpha_i(k+1) = p_i(k)$, $p_i(0) \in [0, 1]$, $i = \overline{1, N}$. Obviously, $p_0(k) \equiv 1$, $p_{00}(k) \equiv 0$. Then, Eqs. (1.1) take the form

$$p_i(k+1) = \mu p_i(k) + (1-\mu)(\Lambda_0 \cdot 1 + \Lambda_{00} \cdot 0) + (1-\mu)\sum_{j=1}^{N} \lambda_{ij} p_j(k+1), \quad i = \overline{1, N}. \tag{4.2}$$

Proposition 7. For any $k = 0, 1, \ldots,$ solutions of system (4.2) $p_i(k) \in [0, 1]$.

The proof follows from Corollary 1 to Theorem 2.

We use $M(k) = p_1(k) + p_2(k) + \ldots p_N(k)$ to denote the mathematical expectation of the number of people supporting the Prophet's idea at the kth step. To obtain the equation for $M(k)$, we add the terms $(1-\mu)\lambda p_i(k+1)$ to both sides of Eqs. (4.2) and sum them. We have

$$M(k+1) + (1-\mu)\lambda M(k+1) = \mu M(k) + N(1-\mu)\Lambda_0 + N(1-\mu)\lambda M(k+1)$$

or

$$M(k+1) = \frac{\mu M(k) + N(1-\mu)\Lambda_0}{1 - (1-\mu)(1 - \Lambda_0 - \Lambda_{00})}. \tag{4.3}$$

We use $\beta(k) = M(k)/N \in [0, 1]$ to denote the fraction of people that share the Prophet's idea. Then, we can rewrite Eq. (4.3) in new variables

$$\beta(k+1) = \frac{\mu\beta(k) + (1-\mu)\Lambda_0}{1 - (1-\mu)(1 - \Lambda_0 - \Lambda_{00})} = \frac{\mu\beta(k) + (1-\mu)\Lambda_0}{\mu + (1-\mu)(\Lambda_0 + \Lambda_{00})}. \tag{4.4}$$

Suppose

$$q = \frac{\mu}{\mu + (1-\mu)(\Lambda_0 + \Lambda_{00})} < 1, \quad b = \frac{(1-\mu)\Lambda_0}{\mu + (1-\mu)(\Lambda_0 + \Lambda_{00})} < 1. \tag{4.5}$$

In this case, the recurrent equation (4.4) takes the form $\beta(k+1) = q\beta(k) + b$, the solution of which is $\beta(k) = \beta(0)q^k + b(1 - q^k)/(1 - q)$ or

$$\beta(k) = \frac{b}{1-q} + \left[\beta(0) - \frac{b}{1-q}\right] q^k. \tag{4.6}$$

It obviously follows from (4.5) and (4.6) that $\beta(k) \underset{k\to\infty}{\to} b/(1-q) = \Lambda_0/(\Lambda_0 + \Lambda_{00}) \le 1$. Note that function (4.6) is increasing for $\beta(0) < b/(1-q) = \Lambda_0/(\Lambda_0 + \Lambda_{00}) \le 1$ and monotonically decreasing for $\beta(0) > \Lambda_0/(\Lambda_0 + \Lambda_{00})$. Thus, if the Prophet is more convincing than the False Prophet ($\Lambda_0 > \Lambda_1$), the fraction of his/her followers $\beta(\infty) > 0.5$. In particular, if $0 < \Lambda_0 \le 1$, $\Lambda_{00} = 0$, then $\beta(\infty) = 1$; i.e., *everyone* who heard him speaking will follow the Prophet. The growth rate of the number of the Prophet's followers depends on their individualism coefficient μ. Thus, for $\mu = 0$, the value $\beta(1) = \Lambda_0/(\Lambda_0 + \Lambda_{00})$. In other words, just for *one* miracle showed by him all atheists ($\beta(0) = 0$) become believers to a reasonable extent and for $\Lambda_{00} = 0$ they become absolute followers of the Prophet. We put $\mu = 0.5$. Suppose $\beta(0) = 0$. Then, we have the inequality $\Lambda_0/(\Lambda_0 + \Lambda_{00}) - \beta(k) \le 10^{-n} \Leftrightarrow k \ge -(n + \log\Lambda_0 - \log(\Lambda_0 + \Lambda_{00}))/\log q$. We put $\Lambda_0 = 0.9$; $\Lambda_{00} = 0$, then $k \ge n/\log 1.9 \approx 3.59n$. Hence, after seven steps (disclosed miracles), 99% of the people are ready to follow the Prophet.

CONCLUSIONS

We proposed a multistep model of the collective behavior of people that generalizes Krasnoshchekov's static model. The model has the form of a linear homogeneous system of difference equations. We studied the mathematical properties of the solutions of this system. In particular, we showed that for a sufficiently large number of stages of an exchange of opinions the group members come to one decision given by the initial conditions. We gave examples of solving two problems (a) on the competition of two groups in parliament and (b) on the influence of the mass media on what people think about an issue.

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